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NUMERICAL ANALYSIS OF FREE VIBRATIONS OF A BEAM WITH OSCILLATORS

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The problem of free vibrations of a beam with free ends of variable cross section and mass, from which point masses (oscillators) are suspended by bars, is considered. It is shown that parametric resonances can occur in this oscillating system. Numerical examples showing the efficiency of the calculation method proposed are given.

Key words: *free vibrations, beam of variable cross section and mass, oscillators.*

Introduction. The eigenvalue problems for the case where the solutions of the corresponding equations are smooth functions are considered in [1]. Some mathematical physics problems, however, lead to eigenvalue problems with piecewise-smooth functions (see below). In the present paper, the results of [1] are generalized to problems with piecewise-smooth functions. The error of the method proposed is estimated in [2]. Longitudinal vibrations of a bar are considered in [3]. The present paper deals with transverse vibrations of a beam with oscillators. The Fortran codes are given in [4].

1. Formulation of the Problem. We consider a beam ($0 \leqslant x \leqslant a$) with free ends. At the points $x = x_k$ ($k = 1, 2, \ldots, n$) of the beam, point masses m_k (oscillators) are suspended by weightless bars of length l_k (parallel to the beam axis at the initial time) in such a manner that a bar with a mass rotating through a small angle relative to the tangent to the beam axis at the suspension point generates a moment proportional to the rotation angle with a coefficient c_k, which tends to bring the mass to the initial position.

We assume that the bars with masses are rigid bodies and the beam is an elastic body with a flexural rigidity $EI_x(x)$ and mass per unit length of the beam $m(x)$. The vibrations are assumed to be infinitely small, so that the point masses perform vibrations in the plane normal to the neutral axis of the beam. We derive the free-vibration equations for this mechanical system.

First, we consider the forces exerted on the beam by a bar with a mass. If the angle φ is small so that the quantities φ^2 can be ignored, then $\sin\varphi \approx \varphi$ and $\cos\varphi \approx 1$;, hence, the mass performs oscillations in the plane normal to the neutral axis of the beam. It follows that the reaction force acting on the beam at the suspension point $x = x_k$ is perpendicular to the beam axis.

Thus, the beam is subjected to the load

$$f(x,t) = M\delta'(x - x_0) + R\delta(x - x_0).$$

We assume that the moment $-M$ acting on a bar with a mass is proportional to the angle between the bar and the tangent to the neutral axis at the suspension point and is directed in such a manner that it tends to bring the mass back to the initial position.

Denoting the deflection by $v(x)$, we write the beam bending equation as

$$(EI_x v'')'' = M\delta'(x - x_0) + R\delta(x - x_0).$$

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Introducing the inertia forces into the beam dynamic equation, we obtain

$$\frac{\partial^2}{\partial x^2}\Big(EI_x\,\frac{\partial^2 v}{\partial x^2}\Big)+m(x)\,\frac{\partial^2 v}{\partial t^2}=R\delta(x-x_0)+M\delta'(x-x_0),$$

where $m(x)$ is the mass per unit length.

Next, we obtain geometrical relations. Let z_0 be the amplitude of the oscillator, i.e., the distance from the mass and the x axis (neutral axis of the beam in the initial state) and y_0 be the amplitude of the suspension point. Then, we have

$$(z-y_0)/l_0=\sin\varphi\approx\varphi,\qquad z_0=y_0+l_0\varphi,$$

for the right location of the oscillator and

$$(z-y_0)/l_0=\sin(\pi-\varphi)=\sin\varphi\approx\varphi,\qquad z_0=y_0+l_0\varphi$$

for the left location of the oscillator.

Taking into account the sign of φ (the positive angle φ is counted anticlockwise), we obtain

$$z_0=y_0\mp l_0\varphi \tag{1.1}$$

(the plus and minus signs refer to the oscillator located on the left and on the right, respectively).

The equations of motions of the oscillator about the suspension point are written as

$$m_0l_0^2\ddot{\varphi}=-M\pm m_0\ddot{y}_0l_0.$$

Here $-M$ is the moment of forces exerted on the beam by the oscillator, $\pm m_0\ddot{y}_0l_0$ is the moment of inertia forces about the point $x=x_0$ (the plus and minus signs refer to the oscillator located on the right and on the left, respectively). If $\ddot{y}_0>0$, the inertia force is directed upward and generates a positive moment for the right location of the oscillator and a negative moment for its left location.

From (1.1), we obtain $\ddot{z}_0=\ddot{y}_0\mp l_0\ddot{\varphi}$. Consequently, $-m_0\ddot{y}_0l_0=-m_0\ddot{z}_0l_0\mp m_0l_0^2\ddot{\varphi}$. The equation $m_0\ddot{z}_0=-R$ is the equation of motion of the center of mass. Thus, we have

$$m_0l_0^2\ddot{\varphi}=-M\mp Rl_0+m_0l_0^2\ddot{\varphi},$$

i.e.,

$$M=\mp Rl_0 \tag{1.2}$$

(the plus and minus signs refer to the oscillator located on the left and on the right, respectively).

The problem considered should be supplemented with the free-vibration equation of the beam with oscillators. We assume that the bar with a mass is subjected to a moment proportional to the angle $\varphi-y'(x_0)$, which tends to bring the mass back:

$$M=c_\varphi(\varphi-y'(x_0))=c_\varphi(z-y_0\mp l_0y'(x_0))/(\mp l_0).$$

Then,

$$R=\mp M/l_0=c_\varphi(z-y_0\mp l_0y'(x_0))/l_0^2.$$

Using the equation of motion of the oscillator $m_0\ddot{z}_0=-R$, we obtain the free-vibration equation for the oscillator

$$-\lambda z_0=-\lambda_0(z_0-y_0\mp l_0y'(x_0)),\qquad \lambda_0=c_\varphi/(m_0l_0^2). \tag{1.3}$$

We derive the equations governing free vibrations of the beam

$$\frac{d^2}{dx^2}\Big(EI_x\,\frac{d^2y}{dx^2}\Big)=\lambda my+R\delta(x-x_0)+M\delta'(x-x_0), \tag{1.4}$$

where

$$R\delta(x-x_0)+M\delta'(x-x_0)=R(\delta(x-x_0)\mp l_0\delta'(x-x_0))$$

$$=c_\varphi(z-y_0\mp l_0y'(x_0))(\delta(x-x_0)\mp l_0\delta'(x-x_0))/l_0^2.$$

For n oscillators, we obtain the equations of free vibrations

$$\frac{d^2}{dx^2}\Big(EI_x\,\frac{d^2y}{dx^2}\Big)=\lambda my+\lambda\sum_{k=1}^{n}m_kz_k(\delta(x-x_0)-l_k\delta'(x-x_0)); \tag{1.5}$$

$$-\lambda z_k=\lambda_k(y_k-z_k+l_ky_k'),\qquad k=1,2,\ldots,n. \tag{1.6}$$

Here $l_k>0$ if the oscillator is on the right and $l_k<0$ if it is on the left,

$$EI_xy''\Big|_{\substack{x=0\\x=a}}=0; \tag{1.7}$$

$$(EI_xy'')'\Big|_{\substack{x=0\\x=a}}=0. \tag{1.8}$$

Equations (1.2)–(1.8) give the desired formulation of the problem of free vibrations of a beam with oscillators.

2. Integral Equation. Let $p(x)\equiv EI_x$. The solvability conditions of Eq. (1.5) are

$$\lambda\int_0^a m(\xi)y(\xi)\,d\xi+\lambda\sum_k m_kz_k=0 \tag{2.1}$$

(the sum of the forces applied to the beam vanishes) and

$$\lambda\int_0^a \xi m(\xi)y(\xi)\,d\xi+\lambda\sum_k m_kz_k(x_k+l_k)=0 \tag{2.2}$$

(the sum of the moments applied to the beam vanishes).

We introduce the Green function $\hat U(x,\xi)$ as the solution of the problem

$$\frac{d^2}{dx^2}\,p(x)\,\frac{d^2}{dx^2}\,\hat U(x,\xi)+\hat c_0(\xi)+x\hat c_1(\xi)=\delta(x-\xi); \tag{2.3}$$

$$p(x)\,\frac{d^2}{dx^2}\,\hat U(x,\xi)\Big|_{\substack{x=0\\x=a}}=0; \tag{2.4}$$

$$\frac{d}{dx}\,p(x)\,\frac{d^2}{dx^2}\,\hat U(x,\xi)\Big|_{\substack{x=0\\x=a}}=0 \tag{2.5}$$

subject to the orthogonality condition for and absolutely rigid displacement

$$\int_0^a m(x)\hat U(x,\xi)\,dx=0,\qquad \int_0^a xm(x)\hat U(x,\xi)\,dx=0. \tag{2.6}$$

The functions $\hat c_0(\xi)$ and $\hat c_1(\xi)$ are chosen so that the system of forces applied to the beam is in equilibrium.

Thus, we have two elastic systems (1.5)–(1.8) and (2.3)–(2.6). Using Betti's reciprocal theorem, we obtain the integral representation of the solution

$$y(x)=\lambda\int_0^a \hat U(x,\xi)m(\xi)y(\xi)\,d\xi+\lambda\sum_{k=1}^{n}m_kz_k(\hat U(x,x_k)+l_k\hat U_\xi'(x,x_k))+c_2x+c_1. \tag{2.7}$$

The constants c_1 and c_2 are chosen so as to satisfy the solvability conditions (2.1) and (2.2). Thus, we have two relations for determining c_1 and c_2:

$$\int_0^a m(x)(c_2x+c_1)\,dx=-\sum_{k=1}^{n}m_kz_k\equiv f_1,$$

$$\int_0^a xm(x)(c_2x+c_1)\,dx=-\sum_{k=1}^{n}m_kz_k(x_k+l_k)\equiv f_2.$$

Consequently,

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where $\alpha_{11} = \int\limits_0^a m(x)\,dx$, $\alpha_{12} = \alpha_{21} = \int\limits_0^a xm(x)\,dx$, and $\alpha_{22} = \int\limits_0^a x^2 m(x)\,dx$.

Let $\beta = \alpha^{-1}$, i.e.,

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1}.$$

Then,

$$c_1 = \beta_{11} f_1 + \beta_{12} f_2, \qquad c_2 = \beta_{21} f_1 + \beta_{22} f_2,$$

where $\beta_{11} = \alpha_{12}/\det\alpha$, $\beta_{21} = \beta_{12} = -\alpha_{12}/\det\alpha$, $\beta_{22} = \alpha_{11}/\det\alpha$, and $\det\alpha = \alpha_{11}\alpha_{22} - \alpha_{12}^2$. Hence, we have,

$$c_2 x + c_1 = (\beta_{21} f_1 + \beta_{22} f_2)x + \beta_{11} f_1 + \beta_{12} f_2$$

$$= (\beta_{21} x + \beta_{11})\Big(-\sum_{k=1}^{n} m_k z_k\Big) + (\beta_{22} x + \beta_{12})\Big(-\sum_{k=1}^{n} m_k z_k (x_k + l_k)\Big).$$

Thus, the amplitudes $y(x)$ and $z_1, \ldots, z_n$ are determined from the system of integro-algebraic equations

$$y(x) + (\beta_{21} x + \beta_{11}) \sum_{k=1}^{n} m_k z_k + (\beta_{22} x + \beta_{12}) \sum_{k=1}^{n} m_k z_k (x_k^* + l_k)$$

$$= \lambda \int\limits_0^a \hat{U}(x,\xi) m(\xi) y(\xi)\,d\xi + \lambda \sum_{k=1}^{n} m_k z_k (\hat{U}(x, x_k^*) + l_k \hat{U}'_\xi(x, x_k^*)); \tag{2.8}$$

$$-\lambda z_k = \lambda_k (y_k - z_k + l_k y'_k), \qquad k = 1, 2, \ldots, n \tag{2.9}$$

(x_k^* is the suspension point of the kth oscillator).

3. Structure of the Finite-Dimensional Problem. Differentiating (2.8), we obtain the supplementary relations for $y'(x)$:

$$y'(x) = -\beta_{21} \sum_{k=1}^{n} m_k z_k - \beta_{22} \sum_{k=1}^{n} m_k z_k (x_k^* + l_k) + \lambda \int\limits_0^a \hat{U}'_x(x,\xi) m(\xi) y(\xi)\,d\xi$$

$$+ \lambda \sum_{k=1}^{n} m_k z_k (\hat{U}'_x(x, x_k^*) + l_k \hat{U}''_{\xi x}(x, x_k^*)). \tag{3.1}$$

Calculating the integral terms in (2.8) and (3.1) by the quadrature formula [3], we arrive at the finite-dimensional problem

$$E \begin{pmatrix} Y \\ Y' \\ z \end{pmatrix} = \lambda D \begin{pmatrix} Y \\ Y' \\ z \end{pmatrix}.$$

Here $Y = (y(x_1), \ldots, y(x_N))^{\mathrm{t}}$ is the vector of the values of the eigenfunction at the interpolation nodes, $Y' = (y'(x_1^*), \ldots, y'(x_n^*))^{\mathrm{t}}$, $z = (z_1, \ldots, z_n)^{\mathrm{t}}$,

$$E = \begin{pmatrix} (N \times N) & (N \times n) & (N \times n) \\ I_N & 0 & \hat{\beta} \\ (n \times N) & (n \times n) & (n \times n) \\ 0 & I_n & \beta^* \\ (n \times N) & (n \times n) & (n \times n) \\ J & -L\Lambda & \Lambda \end{pmatrix}, \tag{3.2}$$

I_N and I_n are the unit matrices of order $N \times N$ and $n \times n$, respectively, $\hat{\beta}_{pk} = (\beta_{21}x_p + \beta_{11})m_k + (\beta_{22}x_p + \beta_{12})m_k(x_k^* + l_k)$, $\beta_{21} = \beta_{12}$ and, hence, $\hat{\beta}_{pk} = \beta_{12}(x_p + x_k^*)m_k + \beta_{11}m_k + \beta_{22}x_p x_k^* m_k + \beta_{22}x_p m_k l_k + \beta_{12}m_k l_k$, β^* is an $n \times n$ matrix with equal columns $\beta_{21}(m_k + \beta_{22}m_k(x_k^* + l_k))$ $(k = 1, 2, \ldots, n)$, J is an $n \times N$ matrix whose kth row $(k = 1, 2, \ldots, n)$ has the only nonzero $j(k)$th component equal to $-\lambda_k$ [$j(k)$ is an integer function that relates the oscillator number k to the node number of the mesh], $L = \operatorname{diag}(l_1, \ldots, l_n)$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$,

$$D = \begin{pmatrix} (N \times N) & (N \times n) & (N \times n) \\ A & 0 & U \\ (n \times N) & (n \times n) & (n \times n) \\ A_x & 0 & U_x \\ (n \times N) & (n \times n) & (n \times n) \\ 0 & 0 & I_n \end{pmatrix},$$

A is an $N \times N$ matrix, $A_{pk} = c_k \hat{U}(x_p, x_k)$, $\hat{U}$ is the Green function, c_k are the coefficients of the quadrature formula [3], U is an $N \times n$ matrix, $U_{pk} = m_k(\hat{U}(x_p, x_k^*) + l_k \hat{U}'_\xi(x_p, x_k^*))$, x_k^* are the suspension points of the oscillators, $\hat{U}'_\xi$ is the derivative of the Green function with respect to the second argument, A_x is an $n \times N$ matrix, $A_{xpk} = c_k \hat{U}'_x(x_p^*, x_k)$, x_p^* are the suspension points of the oscillators, $\hat{U}'_x$ is the derivative of the Green function with respect to the first argument, U_x is an $n \times n$ matrix, $U_{xpk} = m_k(\hat{U}'_x(x_p^*, x_k^*) + l_k \hat{U}''_{\xi x}(x_p^*, x_k^*))$, and $\hat{U}''_{\xi x}$ is the second derivative of the Green function.

Thus, one needs a computer code to calculate the Green functions

$$\hat{U}(x, \xi) = \begin{cases} (\boldsymbol{f}_2(\xi), \hat{A}\boldsymbol{f}_1(x)), & x \leqslant \xi, \\ (\boldsymbol{f}_2(x), \hat{A}\boldsymbol{f}_1(\xi)), & x \geqslant \xi \end{cases}$$

and its derivatives $\hat{U}_x$, $\hat{U}_\xi$, and $\hat{U}_{\xi x}$ (the four-component vectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ and the 4×4 matrix $\hat{A}$ are constructed from 8 arrays and 8 constants calculated using the code).

The matrices U, A_x, and U_x are calculated using the Green functions constructed above.

Writing out the matrices E and D, we note that the structure of the matrix E is similar to that of the matrix E for the longitudinal vibrations of a bar [3] and, hence, it can be inversed analytically. It follows from [3] that one should inverse the matrix $\Lambda - J^* m^*$, where

$$J^* m^* = (J - L\Lambda) \begin{pmatrix} \hat{\beta} \\ \beta^* \end{pmatrix} = J\hat{\beta} - L\Lambda\beta^*.$$

Further, we obtain

$$J\hat{\beta} = -\Lambda \begin{pmatrix} \beta_{J(1),1} & \beta_{J(1),2} & \cdots & \beta_{J(1),n} \\ \beta_{J(2),1} & \beta_{J(2),2} & \cdots & \beta_{J(2),n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{J(n),1} & \beta_{J(n),2} & \cdots & \beta_{J(n),n} \end{pmatrix} = -\Lambda\hat{\beta}.$$

This matrix is obtained from $\hat{\beta}$ by replacing x_p by x_p^* (i.e., suspension points of the oscillators). To inverse this matrix, we consider the inversion of the matrix $I_n + \hat{m}$ for longitudinal vibrations of the bar [3]:

$$\hat{m} = \frac{1}{l} \begin{pmatrix} m_1 & m_2 & \ldots & m_n \\ m_1 & m_2 & \ldots & m_n \\ \ldots & \ldots & \ldots & \ldots \\ m_1 & m_2 & \ldots & m_n \end{pmatrix} = \frac{1}{l} \begin{pmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} (m_1 \ m_2 \ \ldots \ m_n).$$

This matrix has a single eigenvector $(1, 1, \ldots, 1)^{\mathrm{t}}$ and a corresponding eigenvalue $(1/l)\sum m_i$. The matrix $\hat{m}$ is a projector (with accuracy to a scalar multiplier). The matrix $\hat{m}$ transforms any vector x to a vector collinear to $(1, 1, \ldots, 1)^{\mathrm{t}}$.

Thus, we have

$$(I_n + \hat{m})^{-1} = I_n - \frac{1}{1 + (1/l)\sum m_i}\,\hat{m},$$

since $\hat{m}^2 = \Big((1/l)\sum m_i\Big)\hat{m} = \lambda\hat{m}$.

Denoting the eigenvalue of the matrix $\hat{m}$ (n is multiple) by λ, we obtain

$$(I_n + \hat{m})^{-1} = I_n - \frac{1}{\lambda + 1}\,\hat{m}.$$

How can this formula be generalized to the case of a projector to a two-dimensional subspace?

Statement 1. If $\hat{m}^2 = \lambda\hat{m}$, then

$$(I_n + \hat{m})^{-1} = I_n - \frac{1}{\lambda + 1}\,\hat{m}.$$

For transverse vibrations of the beam, one should inverse the $n \times n$ matrix $I + \hat{\beta}_*$ with allowance for $\hat{\beta}_* = \hat{\beta}^{(1)} + \hat{\beta}^{(2)}$, where

$$\hat{\beta}^{(1)}_{pk} = (\beta_{21}(x^*_p + l_p) + \beta_{11})m_k,$$

$$(\hat{\beta}^{(1)})^2 = \lambda_1\hat{\beta}^{(1)}, \qquad \lambda_1 = \sum_{k=1}^{n} m_k(\beta_{21}(x^*_k + l_k) + \beta_{11}); \tag{3.3}$$

$$\hat{\beta}^{(2)}_{pk} = (\beta_{22}(x^*_p + l_p) + \beta_{12})m_k(x^*_k + l_k),$$

$$(\hat{\beta}^{(2)})^2 = \lambda_2\hat{\beta}^{(2)}, \qquad \lambda_2 = \sum_{k=1}^{n} m_k(x^*_k + l_k)(\beta_{22}(x^*_k + l_k) + \beta_{12}); \tag{3.4}$$

$$\hat{\beta}^{(2)}\hat{\beta}^{(1)} = \lambda_{21}\hat{\beta}^{(2)}Q^{-1}, \qquad \lambda_{21} = \sum_{k=1}^{n} m_k(x^*_k + l_k)(\beta_{21}(x^*_k + l_k) + \beta_{12}),$$

$$\hat{\beta}^{(1)}\hat{\beta}^{(2)} = \lambda_{12}\hat{\beta}^{(1)}Q^{-1}, \qquad \lambda_{12} = \sum_{k=1}^{n} m_k(\beta_{22}(x^*_k + l_k) + \beta_{12}), \tag{3.5}$$

$$Q = \operatorname{diag}(x^*_1 + l_1, \ldots, x^*_n + l_n).$$

Using these relations, we obtain

$$(I + \hat{\beta}_*)^{-1} = (I + \hat{\beta}^{(1)} + \hat{\beta}^{(2)})^{-1} = (I + (I + \hat{\beta}^{(1)})^{-1}\hat{\beta}^{(2)})^{-1}(I + \hat{\beta}^{(1)})^{-1}$$

$$= \Big(I + \Big(I - \frac{1}{\lambda_1 + 1}\,\hat{\beta}^{(1)}\hat{\beta}^{(2)}\Big)^{-1}\Big)\Big(I - \frac{1}{\lambda_1 + 1}\,\hat{\beta}^{(1)}\Big),$$

$$\Big(I - \frac{1}{\lambda_1 + 1}\hat{\beta}^{(1)}\hat{\beta}^{(2)}\Big)^{-1} = \Big(I + \hat{\beta}^{(2)} - \frac{\lambda_{12}}{1 + \lambda_1}\,\hat{\beta}^{(1)}Q\Big)^{-1}.$$

We introduce the notation

$$\hat{\beta}^{(3)} = \hat{\beta}^{(2)} - \frac{\lambda_{12}}{1 + \lambda_1}\,\hat{\beta}^{(1)}Q = \Big(\beta_{22}(x^*_p + l_p) + \beta_{12} - \frac{\lambda_{12}}{1 + \lambda_1}\,(\beta_{21}(x^*_p + l_p) + \beta_{11})\Big)m_k(x^*_k + l_k),$$

$$(\hat{\beta}^{(3)})^2 = \lambda_3\hat{\beta}^{(3)}, \tag{3.6}$$

$$\lambda_3 = \sum_{k=1}^{n}\Big(\Big(\beta_{22}(x^*_k + l_k) + \beta_{12} - \frac{\lambda_{12}}{1 + \lambda_1}\,(\beta_{21}(x^*_k + l_k) + \beta_{11})\Big)m_k(x^*_k + l_k)\Big).$$

Then, $(\ldots)^{-1} = I - \hat{\beta}^{(3)}/(\lambda_3 + 1)$. As a result, we obtain

$$(I + \hat{\beta}_*)^{-1} = \Big(I - \frac{1}{\lambda_3 + 1}\,\hat{\beta}^{(3)}\Big)\Big(I - \frac{1}{1 + \lambda_1}\,\hat{\beta}^{(1)}\Big)$$

$$= I - \frac{1}{\lambda_3 + 1}\,\hat{\beta}^{(3)} - \frac{1}{1 + \lambda_1}\,\hat{\beta}^{(1)} + \frac{\hat{\beta}^{(3)}\hat{\beta}^{(1)}}{(\lambda_3 + 1)(\lambda_1 + 1)}.$$

The quantities entering this formula are determined in (3.3)–(3.6). We introduce the notation

$$\lambda_{31} = \sum_{q=1}^{n} m_q (x_q^* + l_q)(\beta_{21}(x_q^* + l_q) + \beta_{11}).$$

Then, we have

$$(I + \hat{\beta}_*)^{-1}_{pk} = \delta_{pk} - \frac{m_k}{\lambda_1 + 1}\,(\beta_{21}(x_p^* + l_p) + \beta_{11})$$

$$+ \frac{m_k}{\lambda_3 + 1}\Big(\frac{\lambda_{31}}{\lambda_1 + 1} - x_k^* - l_k\Big)\Big(\beta_{22}(x_p^* + l_p) + \beta_{12} - \frac{\lambda_{12}}{1 + \lambda_1}\,(\beta_{21}(x_p^* + l_p) + \beta_{11})\Big),$$

$$\delta_{pk} = \begin{cases} 1, & p = k, \\ 0, & p \neq k. \end{cases}$$

The matrix E is determined by (3.2). Introducing the notation

$$J^* = (J - L\Lambda), \qquad m^* = \begin{pmatrix} \hat{\beta} \\ \beta^* \end{pmatrix},$$

after the formal replacement ($J \to J^*$, $m \to m^*$ and $N \to N + n$), we find that this matrix is of the same form as the matrix E for longitudinal vibrations of the bar [3]. As a result, we obtain

$$E^{-1} = \begin{pmatrix} I_N + \hat{\beta}\hat{\Lambda}^* J & -\hat{\beta}\hat{\Lambda}^* L\Lambda & -\hat{\beta}\hat{\Lambda}^* \\ \beta^*\hat{\Lambda}^* J & I_n - \beta^*\hat{\Lambda}^* L\Lambda & -\beta^*\hat{\Lambda}^* \\ -\hat{\Lambda}^* J & \hat{\Lambda}^* L\Lambda & \hat{\Lambda}^* \end{pmatrix},$$

$$E^{-1}D = \begin{pmatrix} (I_N + \hat{\beta}\hat{\Lambda}^* JA) - \hat{\beta}\hat{\Lambda}^* L\Lambda A_x & 0 & (I_N + \hat{\beta}\hat{\Lambda}^* J)U - \hat{\beta}\hat{\Lambda}^* L\Lambda U_x - \hat{\beta}\hat{\Lambda}^* \\ A_x & 0 & U_x \\ -\hat{\Lambda}^* JA + \hat{\Lambda}^* L\Lambda A_x & 0 & -\hat{\Lambda}^* JU + \hat{\Lambda}^* L\Lambda U_x + \hat{\Lambda}^* \end{pmatrix},$$

where $\hat{\Lambda}^* = (I + \hat{\beta}_*)^{-1}\Lambda^{-1}$. Interchanging the second and third columns, we interchange the second and third rows to preserve similarity. As a result, we obtain

$$E^{-1}D = \begin{pmatrix} (I_N + \hat{\beta}\hat{\Lambda}^* JA) - \hat{\beta}\hat{\Lambda}^* L\Lambda A_x & (I_N + \hat{\beta}\hat{\Lambda}^* J)U - \hat{\beta}\hat{\Lambda}^* L\Lambda U_x - \hat{\beta}\hat{\Lambda}^* \\ -\hat{\Lambda}^* JA + \hat{\Lambda}^* L\Lambda A_x & -\hat{\Lambda}^* JU + \hat{\Lambda}^* L\Lambda U_x + \hat{\Lambda}^* \\ A_x & U_x \end{pmatrix}.$$

The zero last column is not written. The eigenvector of this matrix is $(Y, z, Y')^{\mathrm{t}}$, i.e., the desired 2×2 block matrix occupies the upper left corner.

4. Numerical Results. In the numerical analysis, the equations given above were nondimensionalized. The characteristic mass and length were the mass of the beam without oscillators and the beam length. The characteristic time was the quantity $1/W_{\max}$, where $W_{\max}$ is the characteristic frequency in Hz (maximum frequency considered). Calculations were performed to demonstrate the efficiency of the method proposed and to study the occurrence of the parametric resonance in the complex oscillating system considered.

Example 1. A steel beam of circular cross section is considered: $E = 2.1 \cdot 10^6$ kg/cm^2, $\rho = 7.8/981$ g$\cdot$sec^2/cm^4, $a = 10$ m, and $R = 0.1$ m. In this example and below, the range of calculations is 0–30 Hz, i.e., the characteristic time is 1/30 sec. Parameters that enter the beam-vibration equation have the following dimensionless values: $EI_x = 0.0293461$ and $m = 1.0$. The dimensionless value of the squared circular frequency of the system was calculated to be $\lambda = (2\pi w/w_{\max})^2$.

In Example 1, four oscillators with masses $m_1 = 0.1$, $m_2 = 0.7$, $m_3 = 0.7$, and $m_4 = 0.1$ are suspended at the points $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, and $x_4 = 0.8$ from bars of lengths $l_1 = 0.5$, $l_2 = 0.1$, $l_3 = 0.1$, and $l_4 = 0.5$, respectively. All frequencies of the oscillators are identical: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \pi^2/25 = 0.3948$, i.e., equal to 3 Hz.

The dimensionless values of the squared circular frequency are listed in Table 1. The last column (mesh with 99 nodes) contains the values with six significant figures, and the other columns contain correspondingly rounded-off

TABLE 1

Eigenvalue No.	$N = 9$	$N = 19$	$N = 39$	$N = 79$	$N = 99$
1	0.3724	0.37224	0.37220	0.372196	0.372195
2	0.385826	0.385825	0.385825	0.385824	0.385824
3	0.5476	0.546973	0.54680	0.546760	0.546754
4	1.267	1.2656	1.2653	1.26522	1.26522
5	19	17	17	16.82	16.8000
6	129	121	116	114.8	114.620
7	665	504	451	436	434.111
8	1817	1427	1247	1194	1187.46

TABLE 2

Oscillator No.	$\lambda = 0.372195$		$\lambda = 0.385824$		$\lambda = 0.546754$		$\lambda = 1.265220$	
	$N = 99$	$N = 9$	$N = 99$	$N = 9$	$N = 99$	$N = 9$	$N = 99$	$N = 9$
1	1.000000	1.000000	0.470687	0.463	0.188009	0.18	0.422319	0.43
2	−0.06630089	−0.665	0.383055	0.38301	−0.470454	−0.469	0.287537	0.29
3	−0.00477738	−0.49	−0.608918	−0.608	−0.0248762	−0.0241	0.462628	0.47
4	−0.308042	−0.307	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

TABLE 3

Eigenvalue No.	$N = 9$	$N = 19$	$N = 39$	$N = 79$	$N = 99$
1	11.3	11.0	10.93	10.906	10.9030
2	22.51	22.49	22.46	22.456	22.4547
3	—	—	—	340.7	339.737
4	—	—	—	398	396.732
5	—	—	—	1068	1066.44
6	—	—	—	2632	2639.62
7	—	—	—	—	4820.27
8	—	—	—	—	9162.91

Note. The dashes mean that calculations on the corresponding meshes were not performed.

eigenvalues. Results in Tables 2 and 3 are arranged in a similar manner. The amplitudes of the oscillators obtained for the data of Table 1 are listed in Table 2.

Example 2. A round beam of variable cross section is considered: $E = 2.1 \cdot 10^6$ kg/cm^2, $\rho = 7.8/981$ g$\cdot$sec^2/cm^4, $a = 10$ m ($a = a_1 + a_2 + a_3$), $a_1 = a_3 = 3$ m, $a_2 = 4$ m, and $R = 0.1$ m. The dimensionless values of flexural rigidity are equal to 0.0293461 for the first and third segments and 2.25 for the second segment. The dimensionless values of the mass are equal to 0.666666 for the first and third segments and 1.5 for the second segment. Location of the oscillators, their masses, and the lengths of the bars are the same as in Example 1, whereas the dimensionless frequencies are different: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 100\pi = 986.9600$, i.e., the dimensional value is 150 Hz. The calculated eigenvalues are listed in Table 3.

The results obtained show that the model proposed above can be used to study the occurrence of the parametric resonance in a vibrating beam with free ends of variable cross section and mass from which point masses are suspended by bars.

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